Indian Statistical Institute, Bangalore
B. Math.(Hons.) I Year, Second Semester
Mid-Sem Examination
Probability Theory-II

Time: 3 hours | March 4, 2010 | Instructor: B.Rajeev

Maximum Marks 40

1. The joint density function of X and Y is

$$f(x,y) = \begin{cases} x+y & 0<x<1, 0<y<1 \\ 0 & otherwise \end{cases}$$

   (a) Are X and Y independent?

   (b) Find the density function of X.

   (c) Find $P\{X+Y<1\}$. [2+3+3]

2. If X is uniformly distributed over $(0,1)$ and Y is exponentially distributed with parameter $\lambda = 1$, find the distribution of (a) $Z = X+Y$ and (b) $Z = X/Y$. Assume independence. [4+4]

3. Choose a number X at random from the set of numbers $\{1,2,3,4,5\}$. Now choose a number at random from the subset no larger than X. that is, from $\{1,\cdots,X\}$. Call this second number Y.

   (a) Find the joint mass function of X and Y.

   (b) Find the conditional mass function of X given that $Y=1$.

   (c) Are X and Y independent?Prove your answer [3+3+2]

4. The joint density of X and Y is

$$f(x,y) = c(x^2-y^2)e^{-x} \qquad 0 \le x < \infty, -x \le y \le x$$

   Find the conditional distribution of Y, given $X = x$. [5]

5. Suppose that $F(x)$ is a cumulative distribution function and $X \sim F(x)$. (a)Verify that $G(x) \equiv F^n(x)$, $H(x) \equiv 1-[1-F(x)]^n$ and are also cumulative distribution functions when n is a positive integer.

   (b) Show that there exists random variables Y, Z such that $Y \sim G$ and $Z \sim H$. [3+3]

6. Suppose n people $(n \ge 2)$ are distributed at random along a road L miles long. Let $D \le \frac{L}{n-1}$. Show that the probability that no two people are less than a distance of D miles apart is, $[1-(n-1)D/L]^n$. What if $D > L/(n-1)$? [4]

7. Let $X_1, \cdots, X_n$ be independent and identically distributed random variables having distribution function F and density f. The quantity $M \equiv [X_{(1)} + X_{(n)}]/2$, defined to be the average of the smallest and largest value, is called the midrange. Show that its distribution function is

$$F_M(m) = n \int_{-x}^{m} [F(2m - x) - F(x)]^{n-1} f(x) dx.$$

[5]

8. If X and Y are independent standard normal random variables, determine the joint density function of

$$U = X \qquad V = \frac{X}{Y}$$

Then use your result to show that X/Y has a Cauchy distribution.

[4+4]